A Method for the Study of TE and TM Modes in Waveguides of Very General Cross Section

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Abstract—**A simple method for the study of wave propagation in uniform hollow waveguides of very general cross sections is proposed. The method is based upon the concept of contour lines of electromagnetic field components on a typical cross section of the waveguide and applies when the boundary of the cross section of the guide is a closed curve. Examples show that values of cutoff frequencies can be obtained easily to a useful degree of accuracy.**

I. Introduction

Although the general methods of solving the helmholtz scalar equation, which describes the transverse electric (TE) and transverse magnetic (TM) modes of waveguides, are well developed, especially after the publication of a pioneering work by Lord Rayleigh [1], there is considerable labor involved in carrying out the details of the solution for an arbitrary shaped waveguide [2]–[10]. Guides of certain simple shapes have recently been studied extensively. In this context, readers are referred to two excellent review papers [11], [12] in which various approximate methods currently used in waveguide problems with their relative merits and demerits have been summarized.

A new method is presented in this paper which is designed primarily for solving practical microwave problems and to obtain, in an easy manner, the cutoff frequencies and field distribution of one or more waveguide modes. Many a problem for which the exact analytical solution is very involved can be solved in this way, with an accuracy sufficient for many practical purposes.

II. An Account of the Method and the Derivation of a New Equation

We are interested to find the time-periodic electromagnetic fields which can exist inside an infinitely long, metallic cylinder of arbitrary, but uniform, cross section. The space inside the tube is assumed to be completely filled with a homogeneous dielectric. The medium is assumed to be lossless, isotropic, and homogeneous with electrical parameters, μ, ε.

As usual, take the Z-axis as the direction along which the wave propagates. The cross section of this guide forms a closed curve in the XY-plane (Fig. 1). The shape of the electromagnetic field in a transverse plane at any given

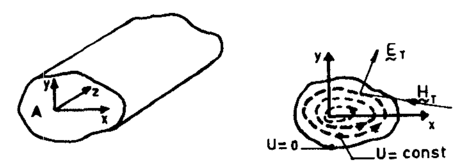

Fig. 1. (a) A waveguide with an arbitrary cross section. (b) The cross sectional contour with magnetic field lines.

time may be visualized by a mapping of the lines of force. The direction of the transverse fields of each eigenwave are independent of the time; in other words, in the transverse planes there exists a time independent field configuration which travels through the waveguides with the phase velocity. The magnetic field lines of the E waves and the electric field of the H waves coincide with the lines of constant E_z and H_z, respectively. Moreover, the transverse electric and transverse magnetic fields are mutually perpendicular to each other.

For TM or TE waves, components of the fields can be derived either from E_z or from H_z. Let the transverse field components be denoted by $\boldsymbol{E}_T$ and $\boldsymbol{H}_T$, respectively. Thus

$$\boldsymbol{E}=\boldsymbol{E}_T+\boldsymbol{E}_z \tag{1}$$

$$\boldsymbol{H}=\boldsymbol{H}_T+\boldsymbol{H}_z. \tag{2}$$

In the case of a TM wave the magnetic field lines coincide with the contour lines for E_z = Const. Denote this family of contour lines by $u(x,y)$ = Const. In the case of a TE wave, the electric field lines coincide with the contour line H_z = Const., and denote this family of contour lines by $v(x,y)$ = Const. It is also known that the magnetic field lines form a closed path surrounding the longitudinal displacement current which is proportional to the longitudinal electric intensity vector $\boldsymbol{E}_z$, and the latter must vanish on the boundaries of the tube. It is clear that the vector $\boldsymbol{E}_T$ is at right angle to the u = Const. curves which are the lines of magnetic force and the vector $\boldsymbol{H}_T$ is therefore along these lines. Lines of constant E_z are shown schematically in Fig. 1(b). In this case the conductor must have one of the lines of constant E_z as its boundary. The lines of electric force (or rather their projections in the XY-plane) are the orthogonal trajectories of the first set of curves. Hence the contour lines $u(x,y)$ = Const. and $v(x,y)$ = Const. are mutually orthogonal. Furthermore, the boundary of the waveguide belongs to the family of magnetic lines and hence the contour lines $u(x,y)$ = Const.

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form a system of nonintersecting closed curves, starting from the outer boundary as one of the lines. Let this family of equipotential curves be denoted by C_u, $0 \leqslant u \leqslant u^*$, so that $C_0 = C$, the boundary of the waveguide, and C_{u^*} coincides with the point(s) at which the maximum $u = u^*$ is attained. Here it has been assumed that u increases inwardly from the outer boundary of the guide.

Let us now set up the integral form of Maxwell's equation. Consider the region in the XY-plane of the waveguide bounded by any closed contour $u(x,y) =$ Const. at any instant t. According to the Maxwell theory, the line integral of the transverse component of the magnetic field around the contour C_u is equal to the time rate of change of the total electric flux through C_u. Thus, one gets Maxwell's postulate in the form

$$\oint_{C_u} H_T ds = \epsilon \frac{\partial}{\partial t} \int_{\Omega_u}\int E_z \, d\Omega \tag{3}$$

where the contour integral is taken around the closed path $u =$ Const., and the double integral is over the region Ω_u enclosed by the same closed contour $u =$ Const.

On differentiating partially with respect to t, and making use of Maxwell's equation in conjuction with (1) and (2), (3) can now be written as

$$\oint_{C_u} \frac{\partial E_z}{\partial n} ds = \frac{1}{c^2} \int_{\Omega_u}\int \frac{\partial^2 E_z}{\partial t^2} d\Omega \tag{4}$$

where $c = 1/\sqrt{\epsilon\mu}$ is the velocity of the disturbance of the dielectric and $\partial/\partial n$ denotes differentiation along the outward normal to the curve. The quantity E_z is in fact a function of space coordinates (x,y,z) and temporal variable t. Assuming harmonic time variations and wave propagation in the positive z direction, then the above equation can be written as shown in [13]

$$\frac{dE_z}{du} \oint_{C_u} \sqrt{\tau} \, ds + k^2 \int_{u^*}^{u} \oint_{C_{u_0}} E_z \frac{ds \, du_0}{\sqrt{\tau}} = 0 \tag{5}$$

where the notations

$$\tau = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

$$k^2 = \frac{\omega^2}{c^2} \tag{6}$$

have been used. Also use has been made of the fact that E_z and its derivatives with respect to u are constant on the contour line $u =$ Const., since E_z depends on u alone.

The problem is thus reduced to solving the integro-differential equation (5) for the field component E_z with Dirichlet's boundary condition. Once this field component has been obtained, all other field components can subsequently be calculated. If we denote cutoff wavenumber by k_0, then the corresponding wavelength is given by

$$\lambda_0 = \frac{2\pi}{k_0}. \tag{7}$$

It is to be noted here that in the case of the TM mode, the magnetic lines of force in a waveguide and the iso-amplitude contour lines for a membrane of the same shape and area of cross section of the waveguide satisfy the same equations and boundary conditions. It will be assumed here that the iso-amplitude contour lines for a freely vibrating membrane in its fundamental mode coincide with the lines of constant deflection for the same membrane under uniformly distributed normal pressure, which can be expressed as

$$\nabla^2 u = \text{Const.} = -2 \ (say) \tag{8}$$

and $u = 0$ on the boundary of the cross section.

III. Solution Procedure

The following relationships are obtained through the application of Green's theorem:

$$\oint_{C_u} \sqrt{\tau} \, ds = -\int_{\Omega_u}\int \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) d\Omega = 2A(u)$$

$$\frac{d}{du} \oint_{C_u} \sqrt{\tau} \, ds = \oint_{C_u} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \frac{ds}{\sqrt{\tau}} = -2 \oint_{C_u} \frac{ds}{\sqrt{\tau}} \tag{9}$$

where $A(u)$ is the area of the regions bounded by the closed curve $u =$ Const. which can be related to the total area A_0 by the relationship

$$A(u) = A_0\left(1 - \frac{u}{u^*}\right). \tag{10}$$

It is to be mentioned here that the relationship (10) holds exactly for a circular and elliptical region and approximately for other regions [14]. Upon differentiation with respect to u and making use of the above relationships, the governing integro-differential equation (5) finally reduces to

$$2(u^* - u)\frac{d^2 E_z}{du^2} - 2\frac{dE_z}{du} + k^2 E_z = 0. \tag{11}$$

In terms of a new independent variable f defined by

$$u^* - u = f^2 \tag{12}$$

the general solution to (11) is

$$E_z = AJ_0(\sqrt{2}\, kf) + BY_0(\sqrt{2}\, kf) \tag{13}$$

and A and B are arbitrary constants and J_0 and Y_0 are Bessel functions of the first and second kinds, respectively. To avoid infinite values of E_z at the point $u = u^*(f = 0)$, it is necessary, when dealing with a hollow simply connected waveguide, to put $B = 0$.

It is interesting to note here that a similar form of solution in polar coordinates has been obtained in [6]. However, the [6] uses two variables (r, θ) in the solution procedure whereas in the present approach, the solution depends only one unknown function $u(x,y)$ or $u(r,\theta)$.

Thus (13) indicates, that for TM modes, we must have

$$J_0(\sqrt{2u^*}\, k) = 0 \tag{14}$$

yielding

$$\sqrt{2u^*}\, k = B_i \tag{15}$$

TABLE I

VALUES OF CUTOFF WAVELENGTH FOR TM_{01} MODE OF AN ELLIPTICAL WAVEGUIDE FOR VARIOUS ASPECT RATIOS

a/b	1.0	1.1	1.2	1.5	2.0	3.0	4.0	5.0	10.0
$\frac{\lambda_0}{a}$	2.6128	2.4855	2.3655	2.0496	1.6525	1.1685	.8962	.7247	.3677

TABLE II

COMPARISON OF NUMERICAL VALUES OF THE CUTOFF WAVELENGTH OF AN ELLIPTICAL WAVEGUIDE BY THE PRESENT METHOD WITH THE EXACT VALUES

Eccentricity	$\frac{\lambda_0}{a}$ by present method	$\frac{\lambda_0}{a}$ exact (Ref.16)
0.20159202	2.5855305	2.5855181
0.59456651	2.3153540	2.3168272
0.95088423	1.0926289	1.1132976

where B_i is the ith root of zero order Bessel function, i.e.,

$$\sqrt{2u^*}\,k = 2.4048, 5.5201, 8.6537, \cdots.$$

Hence, for the cutoff wavenumber of lowest order TM mode, considering the first root of the Bessel function, one obtains

$$k_0 = \frac{2.4048}{\sqrt{2u^*}}. \tag{16}$$

A simple expression has now been obtained for computing cutoff values for the TM mode in a waveguide. In order to judge the degree of accuracy of this expression, several different shaped waveguides will be considered in the next section.

IV. ILLUSTRATION

A. Elliptical Waveguide

As a first example of the above method, consider the case of hollow elliptical waveguides. The exact value of the cutoff frequency of a TM mode in a perfectly conducting and empty elliptical waveguide can only be obtained using complicated Mathieu and associated Mathieu functions [15], [16]. Approximate solutions of this problem have been given by several authors [11], [12].

With the semimajor and semiminor axes of the cross-sectional ellipse being denoted by a and b, respectively, the expression for the lines of magnetic forces, which satisfies (8) is given by

$$u(x,y) = \alpha\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \tag{17}$$

where

$$\alpha = \frac{a^2b^2}{a^2+b^2}. \tag{18}$$

It is clear that $u=0$ on the boundary of the cross section and $u=u^*=a^2b^2/a^2+b^2$ at the center, which is the origin of coordinates.

The cutoff wavelength λ_0 for the TM mode is thus

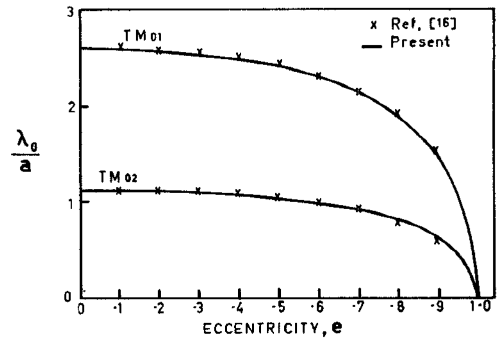

Fig. 2. Variation of cutoff frequency with eccentricity.

TABLE III

CUTOFF WAVELENGTH AND CUTOFF WAVENUMBER FOR AN EQUILATERAL TRIANGULAR WAVEGUIDE

			Error
λ_0	1.005 a	in place of exact value a (Ref. 18)	0.5%
k_0b	7.2144	in place of exact value 7.255 (Ref. 18)	0.5%

obtained as

$$\frac{\lambda_0}{a} = \frac{2\pi}{2.4048}\sqrt{\frac{2}{1+\delta^2}} = \frac{2\pi}{2.4048}\sqrt{\frac{2-2e^2}{2-e^2}} \tag{19}$$

where $\delta = a/b > 1$, and e is the eccentricity of the ellipse.

The numerical values of the parameter λ_0/a for the dominant TM mode (TM_{01}) for the complete spectrum of aspect ratios and for various eccentricities are listed in Tables I and II. Also, in Fig. 2, in order to check the accuracy of the procedure, a few modes over a wide range of eccentricities is given in the mode chart together with those given in [16]. It is evident in the figure that the method described here in a relatively simple fashion leads to excellent agreement with the exact result, indeed the present method gives the graph exactly the same as that of [16].

B. TM Mode of a Waveguide Whose Cross Section is an Equilateral Triangle

As a second example, consider the case of a hollow waveguide with cross section in the form of an equilateral triangle. It is well known that the solution of the Poisson equation (8) in this case yields [17]

$$u(x,y) = \frac{1}{2a}\left(x^3 - 3xy^2 - ax^2 - ay^2 + \frac{4}{27}a^3\right). \tag{20}$$

Obviously $u^* = 2/27a^2$ occurs at the origin of the coordinate system which is the centroid of the triangle of height a.

Calculation for cutoff wavelength λ_0 and cutoff wave number k_0b (where b is the length of the side) is given in Table III.

C. Coaxial Elliptical Waveguide

As the next example, consider the case of a coaxial elliptical waveguide bounded externally and internally by

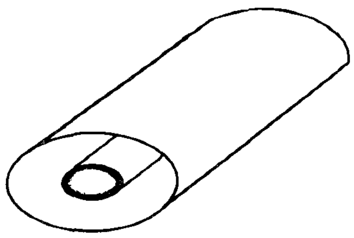

Fig. 3. Coaxial elliptical waveguide

similar ellipses (Fig. 3). In this case the electromagnetic wave propagates in the annular region between two coaxial elliptical conductors and there is zero field outside.

Although, as mentioned earlier, the exact solution of waveguide problems for elliptical regions can be obtained using complicated Mathieu functions, the author believes that this problem has not been discussed so far in the literature, except the case when the two ellipses degenerate into circles [19].

In this case, by consideration of symmetry,

$$u(x, y)=\alpha\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right) \tag{21}$$

where α is given by (18). Further, the similarity condition of two confocal ellipses gives

$$\frac{a_1}{a}=\frac{b_1}{b}=\beta\ (say), \qquad 0<\beta<1. \tag{22}$$

Thus, one gets $u=0$ on the outer boundary of the region and $u=\alpha(1-\beta^2)$ on the inner boundary, or in terms of the variable f given by

$$\alpha-u=f^2 \tag{23}$$

one obtains $f=\sqrt{\alpha}$ on the outer boundary and $f=\beta\sqrt{\alpha}$ on the inner boundary.

Since the second Bessel function in the general waveguide solution given by (13) cannot be excluded in this case, one obtains the following equation by substituting the required boundary conditions:

$$J_0(\sqrt{2\alpha}\,\beta k)Y_0(\sqrt{2\alpha}\,k)-J_0(\sqrt{2\alpha}\,k)Y_0(\sqrt{2\alpha}\,\beta k)=0 \tag{24}$$

which can be written as

$$J_0(\beta\gamma)Y_0(\gamma)-J_0(\gamma)Y_0(\beta\gamma)=0 \tag{25}$$

where

$$\gamma=\sqrt{2\alpha}\,k. \tag{26}$$

It is known from the properties of Bessel functions that the roots of (25) are all real and simple, and that to any positive root γ there corresponds a negative root $-\gamma$.

Consider now the function

$$U_0(\gamma_n\beta)=J_0(\gamma_n\beta)Y_0(\gamma_n)-J_0(\gamma_n)Y_0(\gamma_n\beta). \tag{27}$$

The first seven roots of the above equation are shown in Table IV. For the first simple root, one obtains

$$\sqrt{\frac{2a^2b^2}{a^2+b^2}}\,k_0\beta=0.7632. \tag{28}$$

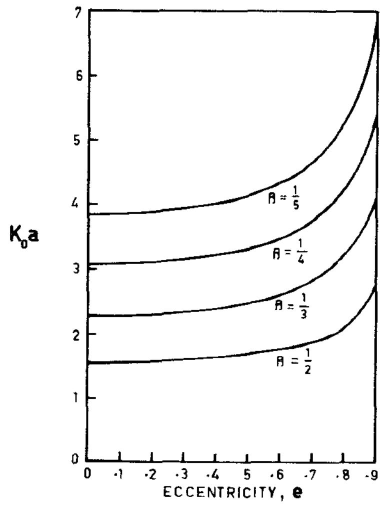

Fig. 4. Variation of cutoff wavenumber with eccentricity for a coaxial elliptical waveguide.

TABLE IV
FIRST SEVEN ROOTS OF [30]

n:	1	2	3	4	5	6	7
$\gamma_n\beta$	.7632	1.5575	2.3479	3.1352	3.9210	4.7073	5.4933

It is interesting to note that if one puts $a=b$ and $a_1=b_1$ so that the two ellipses reduce to circles, then the above equation yields the exact value for the coaxial circular cylindrical waveguide [19] where in this case β becomes the ratio of the two radii of the circles. Equation (28) can also be written in the form

$$k_0a=\frac{0.7632}{\beta}\sqrt{\frac{2-e^2}{2-2e^2}}\ . \tag{29}$$

The numerical values of the cutoff number (k_0a) for various values of β and for a range of values of the eccentricity are shown graphically in Fig. 4.

D. Waveguides having Cross Section in the Form of Semicircle, Semiellipse, and Semiparabola

Finally, let us consider a group of three distinct shaped waveguides having semicircular, semielliptical, and semiparabolic cross sections with respective geometrical dimensions, as shown in Fig. 5. The equations for magnetic contour lines in these three cases are obtained from the knowledge of corresponding torsion functions given in [17], [20], [21] in polar, elliptical, and parabolic coordinates, respectively.

The University of Adelaide's CYBER 173 computer was used to obtain the values of u^* for these three cases. The results of these computations together with the computed values of the cutoff number k_0a are presented in Table V.

V. CONCLUSIONS

A simple and fairly accurate method for the analysis of the hollow waveguide problem with arbitrary shape has been proposed. The essence of the present approach is to

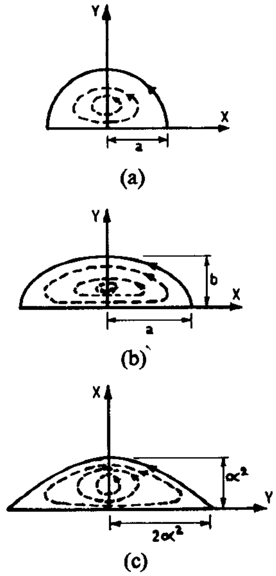

Fig. 5. Waveguides with (a) semicircular, (b) semielliptical, (c) semiparabolic cross section.

TABLE V
NUMERICAL VALUES OF CUTOFF WAVENUMBERS FOR SEMICIRCULAR, SEMIELLIPTIC, AND SEMIPARABOLIC WAVEGUIDES

Type of Waveguide	Value of u*	Value of k_0a	Exact Ref. [19]
Semi-circular	0.1952 a^2	3.8488	3.832
Semi-elliptic with aspect ratio $\frac{b}{a}$ =			
1.0	0.1952 a^2	3.8488	
0.9	0.1656 a^2	4.1786	
0.8	0.1360 a^2	4.6110	
0.7	0.1085 a^2	5.1624	
0.6	0.0822 a^2	5.9310	
0.5	0.0585 a^2	7.0305	
Semi-parabolic with aspect ratio .5	0.0556 a^2 where $a = 2\alpha^2$	7.2115	

reduce the transverse partial differential equation for a longitudinal field component to an ordinary second order differential equation using the concept of contour lines on a typical cross section of the waveguide. Further, it has been shown that if by using the membrane or the torsion analogy or by any of the direct methods of variational calculus the appropriate equation for the family of equipotential lines for any waveguide is known, the problem of determining the cutoff values for that particular domain becomes a very simple affair. The method has been amply illustrated in a selection of practically important problems some of which have not been discussed in literature in the past.

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